The time-varying depth LSW equation for uniformly sloping bottom is

\[ \eta_{tt} - \tan \beta (x \eta_x)_x = h_{0tt}, \]  

(1)

where \( y = \eta(x, t) \) is the free surface elevation, \( y = -h(x, t) \) is the seafloor, and \( h_0(x, t) = H(x) - h_0(x, t) \), i.e., \( h_0(x, t) \) is the time dependent motion of the seabed with respect \( H(x) \). Making the substitution

\[ \zeta = \frac{2 \sqrt{x}}{\sqrt{\tan \beta}}, \]  

(2)

\[ \frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial \zeta^2} - \frac{1}{\zeta} \frac{\partial \eta}{\partial \zeta} = \frac{\partial^2 h_0}{\partial t^2} \]  

(3)

Now define the Hankel transform

\[ \mathcal{H}(\rho, t) = \int_0^\infty \zeta J_0(\rho \zeta) \eta(\zeta, t) \, d\zeta; \]  

(4)

and note that the inverse transform is given by

\[ \eta(\zeta, t) = \int_0^\infty \rho J_0(\rho \zeta) \mathcal{H}(\rho, t) \, d\rho; \]  

(5)

Taking the Hankel transform of (3), one derives that

\[ \frac{\partial^2 \mathcal{H}}{\partial t^2} + \rho^2 \mathcal{H}(\rho, t) = \frac{\partial^2 \mathcal{H}_0}{\partial t^2}, \]  

(6)

where \( \mathcal{H}_0(\rho, t) \) is the Hankel transform of \( \eta_0(x, t) \), i.e., of the seafloor deformation. Now consider a ground deformation of the form

\[ h_0(\zeta, t) = e^{i \omega (\zeta - t)}, \]  

(8)

Then

\[ \int_0^\infty \zeta J_0(\rho \zeta) h_0(\zeta, t) \, d\zeta = \int_0^\infty \zeta J_0(\rho \zeta) e^{i \omega (\zeta - t)} \, d\zeta = -\frac{i \omega e^{-i \omega t}}{(\rho^2 - \omega^2)^{3/2}}, \]  

(9)
with the condition that $\omega/\rho < 1$ and $\omega, \rho > 0$. Taking the second time derivative, one obtains that

$$\int_0^\infty \zeta J_0(\rho \zeta) \frac{\partial^2 h_0}{\partial t^2} d\zeta = \frac{i \omega^3 e^{-i\omega t}}{(\rho^2 - \omega^2)^{3/2}}. \tag{10}$$

Assume that $H(\rho, t) = f(\rho)e^{-i\omega t}$ and then (6) becomes,

$$-\omega^2 f + \rho^2 f = \frac{i \omega^3}{(\rho^2 - \omega^2)^{3/2}}, \tag{11}$$

therefore

$$f(\rho) = \frac{i \omega^3}{(\rho^2 - \omega^2)^{5/2}} \Rightarrow H(\rho, t) = \frac{i \omega^3 e^{-i\omega t}}{(\rho^2 - \omega^2)^{5/2}}. \tag{12}$$

It is known that if $\text{Arg}(\omega'^2) \neq 0$, then

$$\int_0^\infty \frac{\rho J_0(\rho \zeta)}{(\rho^2 - \omega'^2)^{5/2}} d\rho = \frac{1}{3(-\omega'^2)^{3/2}} e^{-\sqrt{-\zeta^2 \omega'^2}} (1 + \sqrt{-\zeta^2 \omega'^2}); \tag{13a}$$

however, it is also known that when $\omega/\zeta < 1$ and $\omega, \zeta > 0$,

$$\int_0^\infty \zeta J_0(\rho \zeta) e^{i\omega \zeta} (1 - i \zeta \omega) d\zeta = \frac{3i \omega^3}{(\zeta^2 - \omega^2)^{5/2}}. \tag{13b}$$

so that combining with (11) one derives that

$$\eta(\zeta, t) = \left( \frac{1}{3} \right) (1 - i \zeta \omega) e^{i\omega(\zeta-t)}. \tag{15}$$

This implies that a particular solution for a motion $h_0(\zeta - t)$ is given by

$$\eta(\zeta, t) = \left( \frac{1}{3} \right) (h - \zeta \frac{\partial h}{\partial \zeta}). \tag{16}$$

(This was confirmed repeatedly and most recently in the file paul/costas.nb on 8/20/01).

Therefore the general solution of the forced equation (1) with a bottom perturbation $h_0(\zeta - t)$ is a combination of the particular solution (16) plus the solution of the homogenous equation $\eta_h(\zeta, t)$. The latter is

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial \zeta^2} - \frac{1}{\zeta} \frac{\partial \eta}{\partial \zeta} = 0, \tag{17}$$

whose solution is

$$\eta_h(\zeta, t) = \int_0^\infty \omega a(\omega) J_0(\omega \zeta) e^{i\omega t} d\omega, \tag{18}$$

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where \( a(\omega) \) to be determined by the requirement that at \( t = 0 \), the sum of the particular solution (15) and the homogenous solution (18) add to zero, corresponding to an unperturned water surface, or

\[
\int_0^\infty \omega a(\omega) J_0(\omega \zeta) d\omega + \left( \frac{1}{3} \right) (h - \zeta \frac{\partial h}{\partial \zeta}) = 0. \tag{19}
\]

Using the orhtogonality of the Bessel functions, one obtains that

\[
a(\omega) = -\frac{1}{3} \int_0^\infty \zeta J_0(\omega \zeta) (h(\zeta, 0) - \zeta \frac{\partial h}{\partial \zeta} |_{t=0}) d\zeta. \tag{20}
\]

To obtain an explicit solution consider an \( h(\zeta, t) = e^{-(\zeta - t)^2} \). Then, (20) becomes

\[
a(\omega) = -\frac{1}{3} \int_0^\infty \zeta J_0(\omega \zeta) (1 + 2\zeta^2) e^{-\zeta^2} d\zeta. \tag{21}
\]

According to MATHEMATICA file IntegralPaulCostas.nb, then

\[
\int_0^\infty \zeta J_0(\zeta) (1 + 2\zeta^2) e^{-\zeta^2} d\zeta = (5/4) e^{-\frac{1}{4}}, \tag{22a}
\]

and

\[
\int_0^\infty \zeta J_0(\omega \zeta) (1 + 2\zeta^2) e^{-\alpha \zeta^2} d\zeta = \frac{1}{4\alpha^3} (4\alpha + 2\alpha^2 - \omega^2) e^{-\frac{\omega^2}{4\alpha}}, \tag{22b}
\]

and when \( \alpha = 1 \),

\[
a(\omega) = \int_0^\infty \zeta J_0(\omega \zeta) (1 + 2\zeta^2) e^{-\zeta^2} d\zeta = \frac{1}{4} (6 - \omega^2) e^{-\frac{1}{4} \omega^2}. \tag{22b}
\]

Therefore the homogenous solution is given by

\[
\eta_h(\zeta, t) = -\frac{1}{4} \frac{1}{3} \int_0^\infty \omega (6 - \omega^2) J_0(\omega \zeta) e^{-\frac{1}{4} \omega^2} e^{i\omega t} d\omega. \tag{23}
\]

At \( \zeta = 0 \), then

\[
\eta_h(\zeta = 0, t) = -\frac{1}{12} \int_0^\infty \omega (6 - \omega^2) e^{-\frac{1}{4} \omega^2} e^{i\omega t} d\omega. \tag{24}
\]

Taking the real part,

\[
\eta_h(\zeta = 0, t) = -\frac{1}{12} \int_0^\infty \omega (6 - \omega^2) e^{-\frac{1}{4} \omega^2} \cos(\omega t) d\omega. \tag{25}
\]

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According to the IntegralPaul/Costas.nb notebook, then

\[ \eta_h(\zeta = 0, t) = -e^{-t^2}[e^{t^2}(4t^2 - 3) - \sqrt{\pi}t(2t^2 - 5)\text{Erfi}(t)]. \] (26)

For the record,

\[ \int_0^\infty \omega(6 - \omega^2)e^{-\frac{1}{4}\omega^2}\sin(\omega t)d\omega = 12e^{-t^2}\sqrt{\pi}t(t^2 - 5). \] (27)

For a further reality check, setting \( t = 0 \) in (23), we obtain

\[ \eta_h(\zeta, t) = -\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\int_0^\infty \omega(6 - \omega^2)J_0(\omega\zeta)e^{-\frac{1}{4}\omega^2}d\omega = \frac{1}{3}(1 + 2\zeta^2)e^{-\zeta^2}, \] (28)

which is the exact opposite of the particular solution.

Setting \( \zeta = 0 \) in (23) we obtain for the shoreline motion,

\[ \eta_h(\zeta, t) = -\left(\frac{1}{12}\right)\int_0^\infty \omega(6 - \omega^2)e^{-\frac{1}{4}\omega^2}\cos(i\omega t)d\omega = -\frac{1}{12}(4 + 8t^2 - 8e^{-t^2}\sqrt{\pi}t^3\text{Erfi}[t]). \] (29)

For small times, we can expand the cosines to obtain a quantitative expression as in IntegralPaulCostas.nb. For any \( t < .4 \), then

\[ -\left(\frac{1}{12}\right)\int_0^\infty \omega(6 - \omega^2)J_0(\omega\zeta)e^{-\frac{1}{4}\omega^2}d\omega = -\left(\frac{4}{45}\right)e^{-\zeta^2}\left(1 + 2\zeta^2\right) + 90t^2(1 - 5x^2 + 2x^4) + 30t^4(-6 + 24x^2 - 15x^4) \] (30)