EXACT SOLUTIONS OF THE IVP OF THE LINEAR SHALLOW–WATER WAVE EQUATION

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(alone for once)

The time–varying depth LSW equation for uniformly sloping bottom is

$$\eta_{tt} - \tan\beta(x\eta_x)_x = h_{0tt}, \tag{1}$$

where $y = \eta(x,t)$ is the free surface elevation, $y = -h(x,t)$ is the seafloor, and $h_0(x,t) = H(x) - h_0(x,t)$, i.e., $h_0(x,t)$ is the time dependent motion of the seabed with respect $H(x)$. Making the substitution

$$\zeta = \frac{2\sqrt{x}}{\sqrt{\tan}\beta}, \tag{2}$$

$$\frac{\partial^2\eta}{\partial t^2} - \frac{\partial^2\eta}{\partial\zeta^2} - \frac{1}{\zeta}\frac{\partial\eta}{\partial\zeta} = \frac{\partial^2 h_0}{\partial t^2} \tag{3}$$

Now define the Hankel transform

$$\mathcal{H}(\rho,t) = \int_0^\infty \zeta J_0(\rho\zeta)\eta(\zeta,t)d\zeta, \tag{4}$$

and note that the inverse transform is given by

$$\eta(\zeta,t) = \int_0^\infty \rho J_0(\rho\zeta)\mathcal{H}(\rho,t)d\rho, \tag{5}$$

Taking the Hankel transform of (3), one derives that

$$\frac{\partial^2\mathcal{H}}{\partial t^2} + \rho^2\mathcal{H}(\rho,t) = \frac{\partial^2\mathcal{H}_0}{\partial t^2}, \tag{6}$$

where $\mathcal{H}_0(\rho,t)$ is the Hankel transform of $\eta_0(x,t)$, i.e., of the seafloor deformation. Now consider a ground deformation of the form

$$h_0(\zeta,t) = e^{i\omega(\zeta-t)}, \tag{8}$$

Then

$$\int_0^\infty \zeta J_0(\rho\zeta)h_0(\zeta,t)d\zeta = \int_0^\infty \zeta J_0(\rho\zeta)e^{i\omega(\zeta-t)}d\zeta = -\frac{i\omega e^{-i\omega t}}{(\rho^2-\omega^2)^{3/2}}, \tag{9}$$

with the condition that $\omega/\rho < 1$ and $\omega, \rho > 0$. Taking the second time derivative, one obtains that

$$\int_0^\infty \zeta J_0(\rho\zeta)\frac{\partial^2 h_0}{\partial t^2}d\zeta = \frac{i\omega^3 e^{-i\omega t}}{(\rho^2-\omega^2)^{3/2}}. \tag{10}$$

Assume that $\mathcal{H}(\rho,t) = f(\rho)e^{-i\omega t}$ and then (6) becomes,

$$-\omega^2 f + \rho^2 f = \frac{i\omega^3}{(\rho^2-\omega^2)^{3/2}}, \tag{11}$$

therefore

$$f(\rho) = \frac{i\omega^3}{(\rho^2-\omega^2)^{5/2}} => \mathcal{H}(\rho,t) = \frac{i\omega^3 e^{-i\omega t}}{(\rho^2-\omega^2)^{5/2}}. \tag{12}$$

It is known that if $Arg(\omega'^2) \neq 0$, then

$$\int_0^\infty \frac{\rho J_0(\rho\zeta)}{(\rho^2-\omega'^2)^{5/2}}d\rho = \frac{1}{3(-\omega'^2)^{3/2}}e^{-\sqrt{-\zeta^2\omega'^2}}(1+\sqrt{-\zeta^2\omega'^2}); \tag{13a}$$

however, it is also known that when $\omega/\zeta < 1$ and $\omega, \zeta > 0$,

$$\int_0^\infty \zeta J_0(\rho\zeta)e^{i\omega\zeta}(1-i\omega\zeta)d\zeta = \frac{3i\omega^3}{(\zeta^2-\omega^2)^{5/2}}. \tag{13b}$$

so that combining with (11) one derives that

$$\eta(\zeta,t) = (\frac{1}{3})(1-i\zeta\omega)e^{i\omega(\zeta-t)}. \tag{15}$$

This implies that a particular solution for a motion $h_0(\zeta - t)$ is given by

$$\eta(\zeta,t) = (\frac{1}{3})(h - \zeta\frac{\partial h}{\partial\zeta}). \tag{16}$$

(This was confirmed repeatedly and most recently in the file paul/costas.nb on 8/20/01).

Therefore the general solution of the forced equation (1) with a bottom perturbation $h_0(\zeta - t)$ is a combination of the particular solution (16) plus the solution of the homogenous equation $\eta_{\rm h}(\zeta,t)$. The latter is

$$\frac{\partial^2\eta}{\partial t^2} - \frac{\partial^2\eta}{\partial\zeta^2} - \frac{1}{\zeta}\frac{\partial\eta}{\partial\zeta} = 0, \tag{17}$$

whose solution is

$$\eta_{\rm h}(\zeta,t) = \int_0^\infty \omega a(\omega)J_0(\omega z)e^{i\omega t}d\omega, \tag{18}$$

where $a(\omega)$ to be determined by the requirement that at $t = 0$, the sum of the particular solution (15) and the homogenous solution (18) add to zero, corresponding to an unperturned water surface, or

$$\int_0^\infty \omega a(\omega) J_0(\omega\zeta) d\omega + (\frac{1}{3})(h - \zeta\frac{\partial h}{\partial \zeta}) = 0. \tag{19}$$

Using the orhtogonality of the Bessel functions, one obtains that

$$a(\omega) = -\frac{1}{3}\int_0^\infty \zeta J_0(\omega\zeta)(h(\zeta,0) - \zeta\frac{\partial h}{\partial \zeta}|_{t=0}) d\zeta. \tag{20}$$

To obtain an explicit solution consider an $h(\zeta, t) = e^{-(\zeta - t)^2}$. Then, (20) becomes

$$a(\omega) = -\frac{1}{3}\int_0^\infty \zeta J_0(\omega\zeta)(1 + 2\zeta^2)e^{-\zeta^2} d\zeta. \tag{21}$$

According to MATHEMATICA file IntegralPaulCostas.nb, then

$$\int_0^\infty \zeta J_0(\zeta)(1 + 2\zeta^2)e^{-\zeta^2} d\zeta = (5/4)e^{-\frac{1}{4}}, \tag{22a}$$

and

$$\int_0^\infty \zeta J_0(\omega\zeta)(1 + 2\zeta^2)e^{-\alpha\zeta^2} d\zeta = \frac{1}{4\alpha^3}(4\alpha + 2\alpha^2 - \omega^2)e^{-\frac{\omega^2}{4\alpha}}, (22b)$$

and when $\alpha = 1$,

$$a(\omega) = \int_0^\infty \zeta J_0(\omega\zeta)(1 + 2\zeta^2)e^{-\zeta^2} d\zeta = \frac{1}{4}(6 - \omega^2)e^{-\frac{1}{4}\omega^2}. \tag{22b}$$

Therefore the homogenous solution is given by

$$\eta_{\mathrm{h}}(\zeta, t) = -\frac{(1}{4)}\frac{(1}{3)}\int_0^\infty \omega(6 - \omega^2)J_0(\omega\zeta)e^{-\frac{1}{4}\omega^2}e^{i\omega t} d\omega. \tag{23}$$

At $\zeta = 0$, then

$$\eta_{\mathrm{h}}(\zeta = 0, t) = -\frac{1}{12}\int_0^\infty \omega(6 - \omega^2)e^{-\frac{1}{4}\omega^2}e^{i\omega t} d\omega. \tag{24}$$

Taking the real part,

$$\eta_{\mathrm{h}}(\zeta = 0, t) = -\frac{1}{12}\int_0^\infty \omega(6 - \omega^2)e^{-\frac{1}{4}\omega^2}\cos(\omega t) d\omega. \tag{25}$$

According to the IntegralPaul/Costas.nb notebook, then

$$\eta_{\rm h}(\zeta=0,t) = -e^{-t^2}[e^{t^2}(4t^2-3) - \sqrt{\pi}t(t^2-5)\text{Erfi}(t)]. \tag{26}$$

For the record,

$$\int_0^\infty \omega(6-\omega^2)e^{-\frac{1}{4}\omega^2}\sin(\omega t)d\omega = 12e^{-t^2}\sqrt{\pi}t(t^2-5). \tag{27}$$

For a further reality check, setting $t=0$ in (23), we obtain

$$\eta_{\rm h}(\zeta,t) = -(\frac{1}{4})(\frac{1}{3})\int_0^\infty \omega(6-\omega^2)J_0(\omega\zeta)e^{-\frac{1}{4}\omega^2}d\omega = \frac{1}{3}(1+2\zeta^2)e^{-\zeta^2}, \tag{28}$$

which is the exact opposite of the particular solution.

Setting $\zeta=0$ in (23) we obtain for the shoreline motion,

$$\eta_{\rm h}(\zeta,t) = -\frac{(1}{12)}\int_0^\infty \omega(6-\omega^2)e^{-\frac{1}{4}\omega^2}\cos(i\omega t)d\omega = -\frac{1}{12}(4+8t^2-8e^{-t^2}\sqrt{\pi}t^3\text{Erfi}[t]). \tag{29}$$

For small times, we can expland the cosines to obtain a quantitative expression as in IntegralPaulCostas.nb. For any $t<.4$, then

$$-(\frac{1}{12})\int_0^\infty \omega(6-\omega^2)J_0(\omega\zeta)e^{-\frac{1}{4}\omega^2}d\omega = -(\frac{}{12})(\frac{4}{45}e^{-\zeta^2}((1+2\zeta^2)+90t^2(1-5x^2+2x^4)+30t^4(-6+24x^2-15x \tag{30}$$